

Measuring Anharmonicity in a Large Amplitude Pendulum

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01.50.Lc, 01.50.Pa, 01.55.+b, 2.40.Hw, 2.40.Yy, 45.50.Dd

The anharmonicity of a large amplitude pendulum is investigated experimentally. A novel technique that allows the detection of high Fourier components as a function of the amplitude for large amplitudes, close to 180° , was developed. It consists in carrying out a Fourier analysis on each half-cycle of the amplitude versus time. The presence of the third and fifth harmonics was detected and the variations of their corresponding Fourier coefficients with amplitude were studied. The experimental setup is inexpensive and simple to implement.

I. INTRODUCTION

The motion of a pendulum is harmonic only for very small angular deflections. At larger amplitudes the nonlinear effect becomes increasingly more important and cannot be neglected.^{1,2,3} Since the precision of modern laboratory equipment, such as photogates and shaft encoders, is very high, the effects of nonlinearity in the motion of the pendulum can be easily detected with very simple and low cost equipment. Incidentally, this circumstance offers a splendid opportunity to explore new and exciting physics⁴ with this very simple and readily available system. In particular, the effect of nonlinearity on the dependence of the period of the pendulum with amplitude has been discussed in several publications in recent years.^{5,6,7,8} On the other hand, the literature on experimental and theoretical studies on the frequency spectral composition of the angular deflection of the pendulum as a function of time, $\theta(t)$, is much more scarce and limited.^{9,10,11} Previous studies have been able to detect only the third harmonic, and reported agreement with the perturbative theoretical prediction up to angles θ_0 less than about 2.2 rad .¹¹ The purpose of this work is to explore, experimentally, the Fourier components of $\theta(t)$ as a function of the maximum amplitude of the pendulum, θ_0 , and to compare these results with different theoretical expectations. To carry out our study, we developed a novel technique for extracting the Fourier coefficient of a signal that has periods (fundamental frequency) that change with time. It consists in carrying out a Fourier analysis on each half-cycle of the amplitude versus time. This technique is discussed in detail in appendix A and is applied to both the experimental deflection as a function of the time and the theoretical solution of the differential equation that describes the motion of the pendulum.

II. THEORETICAL CONSIDERATIONS

In absence of friction, the equation of motion of a physical pendulum is:^{1,2,3}

$$\frac{d^2\theta}{dt^2} + \omega_0^2 \sin(\theta) = 0, \quad (1)$$

where $\omega_0^2 = mgd_{cm} / I$ is the natural frequency of the pendulum in the limit of very small amplitudes. As usual, m represents the mass of the pendulum, I the moment of inertia of the physical pendulum relative to the pivot, d_{cm} the distance from this point to the center of gravity of the physical pendulum, and g the acceleration of gravity. Here, θ is the angular deflection (amplitude) of the pendulum with respect to the vertical that passes through the pivot. Although there is a closed analytical solution of this equation in terms of the Jacobean elliptical functions,³ it is simpler and more convenient to our purpose to solve Eq.(1) using a perturbative approach.^{2,7,10,12} This technique consists of expanding $\sin\theta$ in a Maclaurin series, and then solving Eq.(1), recursively. If the initial conditions are $\theta(0) = \theta_0$ and $\dot{\theta}(0) = 0$, the solution of Eq.(1) is:^{7,8,10,11,12,13}

$$\begin{aligned} \theta(t) \approx & \theta_0 \cos \omega t + \frac{\theta_0^3}{192} (\cos(\omega t) - \cos(3\omega t)) + \\ & + \frac{\theta_0^5}{512} [(17/120) \cos(\omega t) - (1/6) \cos(3\omega t) + (1/40) \cos(5\omega t)] + \dots \end{aligned} \quad (2)$$

with

$$\omega_0^2 \approx \omega^2 \cdot [1 + \theta_0^2 / 8 + 17\theta_0^4 / 1536 + \dots]. \quad (3)$$

Equation (2) can also be written as:

$$\theta(t) = \sum_{n=0}^{\infty} A_{2n+1}(\theta_0) \cos[(2n+1)\omega t], \quad (4)$$

with

$$A_1(\theta_0) = \theta_0 + \theta_0^3 / 192 + (17/120)\theta_0^5 / 512 + \dots, \quad (5)$$

$$A_3(\theta_0) = -[\theta_0^3 / 192 + (1/6)\theta_0^5 / 512 + \dots] = -(A_1^3 / 192)[1 + A_1^2 / 16 + \dots], \quad (6)$$

$$A_5(\theta_0) = (1/40)\theta_0^5 / 512 + \dots = A_1^5 / 20480 + \dots, \quad \text{etc.} \quad (7)$$

Being A_n the Fourier coefficients of the spectral decomposition of $\theta(t)$. As expected, for very small amplitudes ($\theta_0 \approx 0$), the motion of the pendulum is harmonic with frequency ω_0 . As the amplitude increases, the odd harmonic terms become important; so the anharmonic behavior sets in. Therefore, the onset of nonlinear effects produces a variation of the frequency (period) with amplitude (θ_0), and gives rise to odd harmonic terms in $\theta(t)$. To experimentally detect this effect, in principle one could use a pendulum with negligible friction, measure $\theta(t)$ and then carry out a Fourier analysis of $\theta(t)$. In practice, it is not always possible to build a frictionless pendulum. Therefore, each half-cycle of the pendulum will have not only a different amplitude but also a different fundamental frequency, i.e. a different period, that makes the problem of detecting the high harmonic frequencies very difficult.

In the perturbative approach just described, the expansion parameter is θ . Therefore, we would expect that the results deduced from this theory will be valid for angles θ_0 less than about 1 rad ($\approx 57^\circ$) and when the effect of damping are negligible.

Since one is interested in exploring larger amplitudes, where the anharmonic effect is expected to be even larger, this study will also allow us to explore the range of validity of the perturbative solution.

As the amplitude increases, the characteristic velocity of the pendulum also increases. Thus, the effects of viscous and turbulent frictions,⁴ that depend on the velocity and the square of the velocity, respectively, also increase concomitantly. Therefore, for large amplitudes ($\theta_0 \approx \pi$) the effect of friction, in general, cannot be neglected. In this case, the equation of motion of the pendulum can be written as:⁴

$$\frac{d^2\theta}{dt^2} = -\omega_0^2 \sin(\theta) - 2\gamma \frac{d\theta}{dt} - \beta \frac{d\theta}{dt} \left| \frac{d\theta}{dt} \right|, \quad (8)$$

where γ and β are two constants associated with the strength of the viscous and turbulent components of the damping force, respectively. The turbulent component of the friction force, proportional to $\dot{\theta}^2$, is relevant whenever the characteristic Reynolds number of the motion of the pendulum in air is of the order of or larger than 4000,⁴ which in our case was almost always the case, especially at large amplitudes. Equation (8) does not have an analytical solution, but it can be solved numerically. However, there is an approximate expression that describes the amplitude of the pendulum as function of time.⁴ As expected, the turbulent component is relatively more relevant when the amplitude is large, and consequently, the characteristic velocities are also large. As the amplitude decreases, the viscous component dominates, leading to an exponential decrease in the amplitude as a function of time. A simple program for solving this equation using predefined Runge-Kutta methods for solving ODEs of Matlab®^{14,15} was written; the numerical solution is denoted by $\theta_T(t)$. The parameters ω_0 , γ and β were obtained from the fit of the theoretical model (θ_T) to the experimental data of θ versus t . The parameters ω_0 , γ and β are the only adjustable parameters in our model. Of course, ω_0 could also be calculated from the geometry of our pendulum as indicated in Eq. (1). Nonetheless, due to the complex geometry of our pendulum, we obtained this parameter from the small angle oscillation of our pendulum. In this regard, our pendulum was optimized to study the large amplitude effects rather than to measure the value of g with great precision.⁴ Once we obtained the theoretical model (θ_T) that reproduces the experimental data, we carried out a Fourier analysis of each half cycle of this solution, using the procedure discussed in Appendix A. This algorithm yields the amplitude θ_0 and the Fourier coefficients, A_i , associated to each half cycle.

III. THE EXPERIMENT

Our pendulum consists of an aluminum disc, 29.5 cm in diameter and 3 mm thick, suspended at its center. The disc with its fixation hub, weights 967 g. It is fixed to the axis of a shaft encoder as shown in Fig. 1. An extra weight, of 50 or 100 g, can be fastened with a brass bolt placed on the face of the disc, close to its edge, to break the mass symmetry of the system. The disc has several holes, evenly spaced in a radial direction, allowing the distance of the weight to its center to be changed. Also the magnitude of the weight can be modified. Manipulating the position and magnitude of the weight, we could easily regulate the asymptotic period of the pendulum, i.e. the period at very small amplitude. A

Neodymium magnet (diameter 12.7 mm and 6.35 mm thick; grade N40) can be placed at 10 cm from the axis of rotation. Its distance to the disc can be regulated with a bolt. In this manner the friction coefficient (proportional to the velocity) can be changed. An US-Digital¹⁶ S4 optical shaft encoder connected to a computer, with angular resolution of 0.3 degrees, was used to monitor the angular position of the pendulum. Therefore, the estimated error in the determination of the angles was at most 0.5 degrees. Because error bars associated to these uncertainties would be smaller than the size of the symbols used to represent the data, they were not shown in Figs. 2-4.

Due to the very low friction of the shaft encoder and the relatively large mass of the pendulum, the free oscillation lasted more than 250 oscillations. The angles were measured at a rate of 20 Hz, so about 60 data points characterized every period. The mechanical design of our pendulum is similar to the arrangement used in Ref [17].

IV. RESULTS AND DISCUSSION

In Fig. 2 we display $\theta_e(t)$ (diamond symbols), the experimentally measured angles as a function of time, for very large amplitudes ($\theta_0 \approx \pi$). The continuous line in this figure is a single frequency cosine function that has the same period and amplitude as that of the pendulum. The deviation of the experimental results relative to the pure cosine function reveals the presence in the data of higher harmonic components. In Fig. 3 we present the behavior of $\theta_e(t)$ for a longer time, together with the theoretical expectation.

The effects of the nonlinearity, revealed in the lack of isochronism of the pendulum, are clearly seen. The period of the first cycle is 4.2 s, whereas that of the fifth cycle is 3.2 s.

Using the numerical algorithms described in Appendix A, we carried out a Fourier analysis of experimentally measures deflection function, $\theta_e(t)$, for each half cycle. We thus explored the variation of the Fourier coefficients, A_n , for the fundamental frequency and its harmonics, as a function of the amplitude (θ_0). Fig. 4 shows the variation of A_1 (associated to the fundamental frequency) with θ_0 . Similarly, Fig. 5 shows the variation of A_3 and A_5 as a function of the amplitude. In Figs. 4 and 5, the symbols represent the Fourier coefficients extracted from the experimental data $\theta_e(t)$, while the continuous lines are the corresponding predictions from the models proposed. In Fig. 5 we show, with heavy continuous lines, the theoretical expectation of the behavior of A_3 and A_5 versus θ_0 obtained by carrying out a Fourier analysis on the theoretical expression $\theta_T(t)$. This function was obtained by solving Eq.(8) numerically with parameters that reproduce the experimental results, $\theta_e(t)$, as illustrated in Fig.2. We see that this theoretical model does provide an adequate description of the experimental data for all the amplitudes studied. In Fig. 5, we also show (dotted lines) the theoretical expectation obtained using the perturbation approach, Eqs.(4) to (7), that reproduce de data only for $\theta_0 < 2.2$ rad. Similar lack of agreement between the perturbative approach and the experimental results was already visible in pioneer work of Zilio.¹¹ Since in the perturbative approach,^{2,7,8,12} the expansion parameter is θ , our original expectation was that expressions (5) to (7) would break down for some angle θ_0 of order of 1 rad. Remarkably, as can be observed in Figs. 4 and 5, this was not the case and the

perturbative solution reproduced the experimental data quite well even for amplitudes close to 2.2 rad. It is interesting to note that the Fourier coefficients obtained with the perturbative approach, coincided very well with the corresponding coefficients obtained using the half-cycle analysis on the numerical solution of Eq.(1), i.e. $\theta_1(t)$ for the case of no damping. The presence of damping introduces three effects in the solution, variation of amplitude with time, variation of the period relative to the frictionless case, and variation of the dependence of the Fourier coefficients as a function of the amplitude. Fig. 5 illustrates this situation, where we see that only for $\theta_0 > 2.2$ rad, the difference between the theoretical approaches becomes important. For angles close to 180° , the Fourier analysis carried out on the numerical solution of the differential equation (8), provides a much better description of the experimental data than the perturbative approach, carried out up to the fifth order, as we have done here.

In summary, we have built a pendulum to investigate experimentally the behavior of the Fourier coefficients as a function of the amplitude. We were able to detect the third and fifth harmonics and marginally the seventh harmonic, for amplitudes close to 180° . The experimental result correlates well with the results predicted by the model proposed herein, for angular amplitudes up to 3 rad. The range of applicability of the perturbation approach is clearly illustrated ($\theta_0 < 2.2$ rad). The setup used is very low cost, simple to implement and construct. We hope that this experiment can be profitably implemented in intermediate laboratory courses, to illustrate an important consequence of a nonlinear system. This issue is not often discussed in the literature.

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APPENDIX A

Fourier analysis of anharmonic signals - Half-cycle Fourier analysis

In this appendix we summarize algorithm used for carrying out the half-cycle Fourier analysis on both, the experimentally measured angular deflection function, $\theta_e(t)$, and the numerical solution of the differential equation of our pendulum, $\theta_1(t)$, obtained by solving Eq.(8). The basic characteristic of the signal of interest, $\theta(t)$, is that its period and amplitude decrease monotonically with time, as shown in Fig. 3. Since the raw data collected with the data acquisition system, are not evenly spaced out in time, a spline interpolation is carried out using Matlab®. This generates a new set of angles evenly spaced out in time at a rate of about 200Hz. Next, the zero crossing times of the signal are searched. With this last information a set of half cycles or semicycles are defined,

comprising two consecutive zeros and all the values interpolated from the measured angles, within these time intervals, $\theta^{(k)}(t)$. The absolute value of $\theta^{(k)}(t)$ is then applied to each semicycle.

The maximum of each semicycle characterizes its amplitude $\theta_0^{(k)}$. Next, a complete pseudo-cycle is constructed with each semicycle, whose period $T^{(k)}$ is twice the difference between two adjacent zeroes that limit each semicycle. The super-index (k) characterizes each pseudo-cycle. The pseudo-cycle comprises the corresponding semicycle and its reflection:

$$\theta^{(k)}(t) = \begin{cases} \theta^{(k)}(t) & \text{if } 0 \leq t \leq T^{(k)} / 2 \\ -\theta^{(k)}(T^{(k)} - t) & \text{if } T^{(k)} / 2 < t < T^{(k)} \end{cases} \quad (\text{A1})$$

In this manner, from each semicycle (k), we obtain a complete pseudo-cycle, characterized by a vector $\theta_j^{(k)}$ of dimension $N^{(k)}$, representing the angular position of the pendulum at evenly spaced intervals of time. A Fourier analysis is carried out on each pseudo-cycle, and the Fourier coefficients $A(n)$ are calculated as¹⁸:

$$A^{(k)}(n) = \frac{2}{T^{(k)}} \int_0^{T^{(k)}} \theta(t) \cdot \text{Exp}\left(i \frac{2\pi}{T^{(k)}} t\right) \cdot dt \approx \frac{N^{(k)}}{2} \sum_{j=0}^{N^{(k)}-1} \theta_j^{(k)} \cdot \text{Exp}(i 2\pi \cdot n / N^{(k)}). \quad (\text{A2})$$

Here, $i = \sqrt{-1}$, the imaginary unit. As a consequence of the symmetry property given by Eq.(A1), only the odd Fourier coefficients will be different from zero. Using this procedure, implemented by using Matlab^{®15}, we are able to extract, for each half cycle (k), an amplitude, $\theta_0^{(k)}$, a period, $T^{(k)}$, and the Fourier coefficients $A^{(k)}(n)$.

To test the reliability of our method, we use our program on an artificial signal with a well known composition of Fourier coefficients. Let us define a signal function:

$$f_T(t) = A_0 \cdot \exp(-\gamma \cdot t) \cdot [\sin(\phi(t)) + B3 \cdot \sin(3 \cdot \phi(t)) + B5 \cdot \sin(5 \cdot \phi(t)) + B7 \cdot \sin(7 \cdot \phi(t))] \quad (\text{A3})$$

with

$$\phi(t) = \int_0^t \omega(u) \cdot du \quad \text{and} \quad \omega(t) = \omega_0(1 + a_0 \cdot t). \quad (\text{A4})$$

Here ω_0 and a_0 are two constants that generate a signal with variable frequency or period. The constants A_0 , γ , $B3$, $B5$, and $B7$ are chosen to simulate actual data. Thus, the function given by Eq.(A3) represents a signal that has variable frequency and variable Fourier components

$$A^{(1)}(1) = \text{Fundamental frequency amplitude} = A_0 \cdot \exp(-\gamma \cdot t) \quad (\text{A5})$$

$$A^{(1)}(3) = \text{Tird harmonic amplitude} = A_0 \cdot \exp(-\gamma \cdot t) \cdot B3 \quad (\text{A6})$$

and so on.

It is important to note that the signal described by Eq.(A1) is an artificial signal, created to mimic the basic characteristics of a real signal. Also, its Fourier coefficients have a well known dependence on time and amplitude, Eqs.(A5) and (A6), but are not expected to match exactly the actual pendulum data. Therefore, this artificial signal can be used as a benchmark for testing the performance of the half-cycle Fourier analysis proposed in this article.

In order to introduce some randomness into the trail signal, characteristic of any experimental data, we use a simplified version of the Monte Carlo technique. If $Disp$ represents the characteristic dispersion of the data, a new signal is defined:

$$f_{sint}(t) = f_T(t) \cdot [1 + Disp \times Rand_nd], \quad (A7)$$

where $Rand_nd$ is a random number, with *normal distribution*, mean equal to 0 and standard deviation equal to 1. Thus, we have constructed a “synthetic” signal that resembles the signal that we would like to study, but with *known behavior and parameters*, since the values γ , ω_0 , a_0 , B_i and $Disp$ are chosen *a priori*. To test the Fourier analysis method proposed above we introduce the function given by Eq. (A7) into our program and compare the Fourier coefficients thus obtained with the input parameters, defined by Eqs. (A4), (A5) and (A6). In Fig. 6 we present an example of the “synthetic” signal using a $Disp=2\%$.

The results of the Fourier analysis of the “synthetic” signal are summarized in Figs. 7 and 8. We see that for a signal with dispersion of 2%, considerably larger than the dispersion we expect on our actual data, we can reliably obtain the first three odd harmonics with our method. Consequently, Figs. 7 and 8 indicate that the half-cycle Fourier analysis proposed in this work is adequate for detecting the third, fifth and seventh Fourier coefficients of magnitude comparable to the one observed in our actual data, Figs. (4) and (5).

Figure Captions

Fig. 1. Diagram of the experimental setup.

Fig. 2. Angular deflection as a function of time for the free pendulum, when its amplitude is close to 180° . The diamond symbols are the experimental data. The continuous line is a single frequency cosine function that has the same period as that of the pendulum. The deviation of the experimental results with reference to the pure cosine function reveals the presence of higher harmonic components.

Fig. 3. Angular deflection as a function of time for the free pendulum. The diamond symbols are the experimental data and the continuous line is the result of the model including friction, obtained by solving Eq.(8) numerically, with parameters that reproduce the experimental results.

Fig. 4. First Fourier coefficient, A_1 versus amplitude for the real angular deflection of the pendulum. The symbols represent the Fourier coefficients obtained from the experimental data of θ versus t . The dotted line is the prediction of the perturbation approach, given by Eq.(5). The heavy continuous line represents the results of a half-cycle Fourier analysis on the theoretical model, $\theta_T(t)$, obtained by solving Eq.(8) numerically.

Fig. 5. Fourier coefficients versus amplitude for the first two odd harmonics for the real angular deflection of the pendulum. The coefficients associated to *third harmonic*, A_3 , are

referred to the left vertical axis, while those associated to *fifth harmonics*, A_5 , are referred to the right vertical axis. The symbols (open triangle and circles) are the Fourier coefficients obtained from the experimental angular deflection of the pendulum. The dotted line is the prediction of the perturbation approach, given by Eqs. (6) and (7), respectively. The heavy continuous line represents the results of a Fourier analysis on the theoretical model, $\theta_T(t)$, obtained by solving Eq.(8) numerically.

Fig.6. Artificial angular deflection. The thin continuous line represents the artificial angular distribution, Eq.(A3). The square symbols are the synthetic data, including randomness, obtained using expression (A7). The dashed line represents the variation in the angular frequency $\omega(t)$ with time, referred to the right vertical axis, Eq.(A4).

Fig.7. Frequency and first Fourier coefficient as a function of the amplitude for the artificial signal. The continuous curves are theoretical inputs, Eqs.(A4) and (A5). The symbols are the results obtained using our half-cycle Fourier analysis of the “synthetic” signal, given by Eq. (A8) as input and with $Disp=2\%$. This synthetic signal was created to mimic the basic characteristics of a real signal, but is not expected to match exactly the actual pendulum data. The circles represent the Fourier coefficients A_i , referred to the left vertical axis and the rhomboidal symbols are the frequencies, extracted from the artificial signal, referred to the right vertical axis.

Fig. 8. Fourier coefficients as a function of amplitude for the artificial signal. A_i is the Fourier coefficient associated to the i harmonic of the fundamental frequency. The continuous curves are theoretical inputs, obtained using Eq.(A4). The symbols are the results obtained using our half-cycle Fourier analysis of the “synthetic” signal, given by Eq.(A8) with $Disp=2\%$.

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- ¹² L. A. Pipes and L. R. Harvill, *Applied Mathematics for Engineers and physicists*, **3rd** edition (McGaw-Hill, New York, 1970), Vol. **1**, Chap. 22, p.995.
- ¹³ There is an apparent discrepancy between the theoretical results of the perturbation method, Eqs.(2), (3), (5) and (7), presented here and in the work of Fulcher and Davis (ref. 8) and those appearing in the work of Zilio (ref.11). Whereas Fulcher and Davis had indicated multiplication by a large centered dot in $8 \times 64 = 512$ and $24 \times 64 = 1536$, these values appear in Zilio's paper as 8.64 and 24.64. Most likely the notation was changed in the publication process. If these typos are taken into account the two results are the same. We are grateful to one of the reviewers of this article for pointing out this apparent disagreement.
- ¹⁴ Matlab® MathWorks inc., 21 Eliot St. South, Natick, MA 01760.
<http://www.mathworks.com/>
- ¹⁵ Further details of our experiment and a copy of the Matlab program to carry out the Fourier analysis of the experimental results and for solving numerically the differential Eq.(8), can be obtained from <www.fisicarecreativa.com>.
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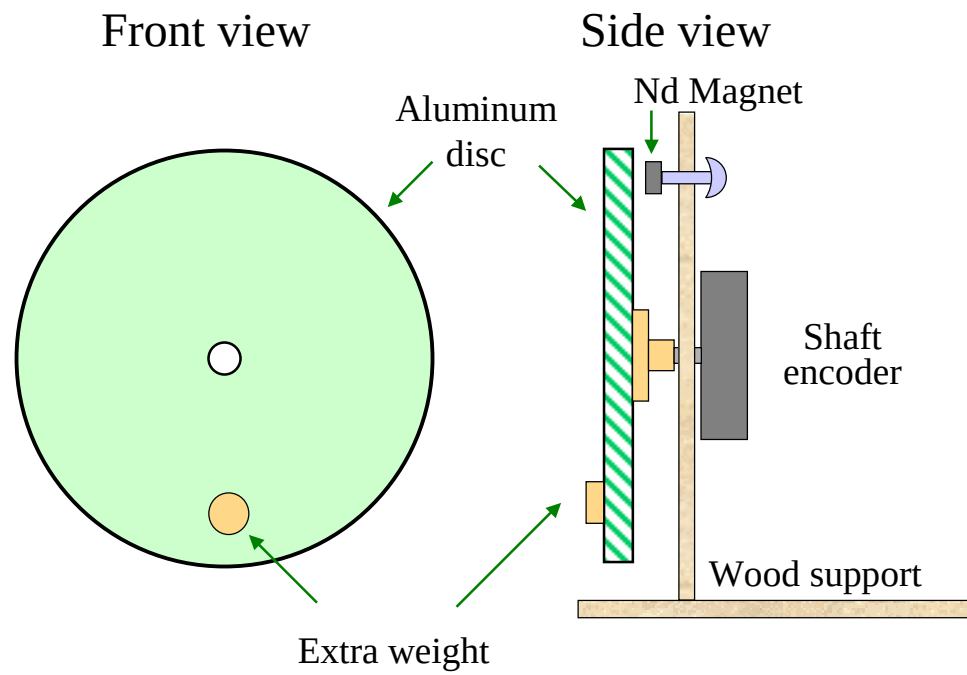


Fig. 1

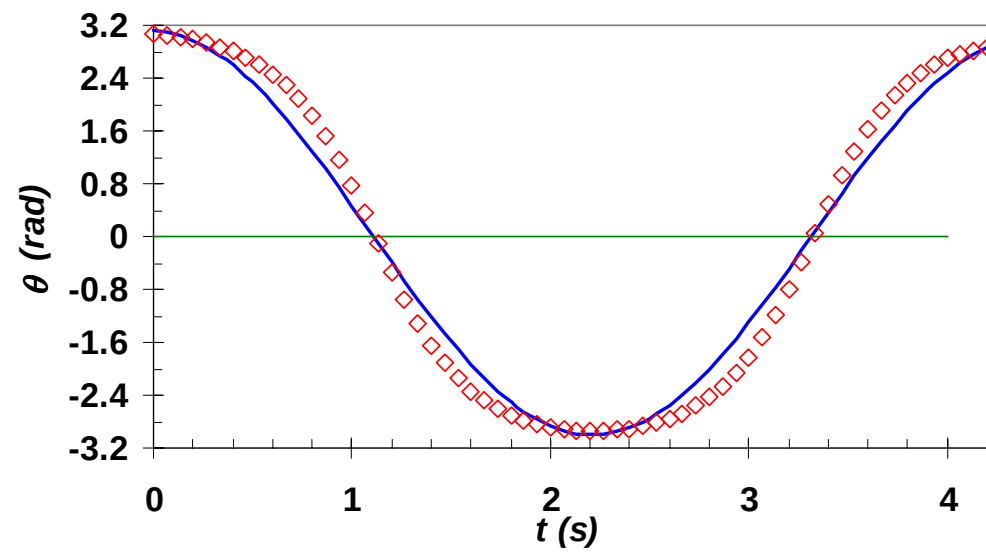


Fig. 2

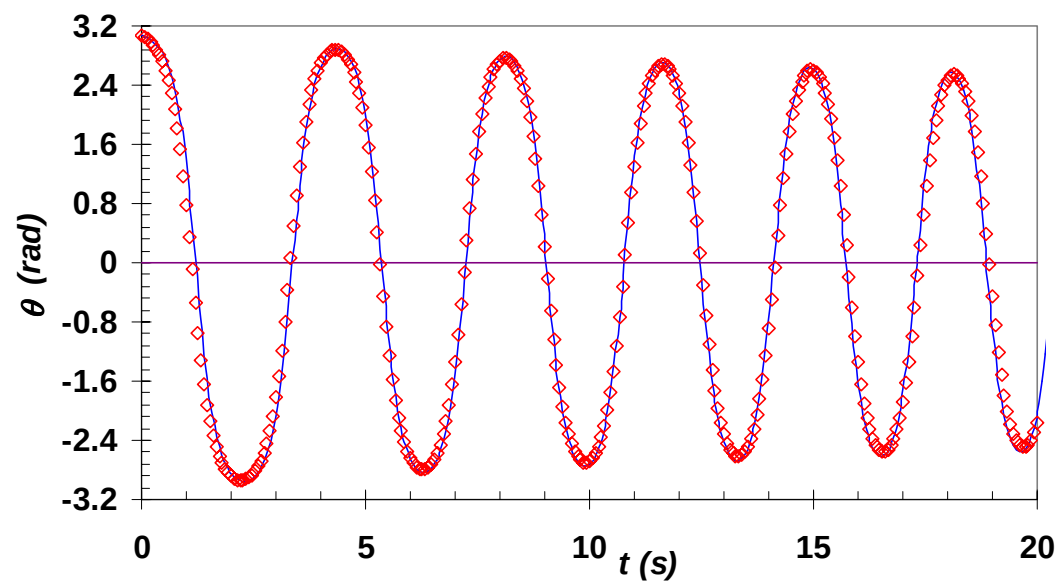


Fig. 3

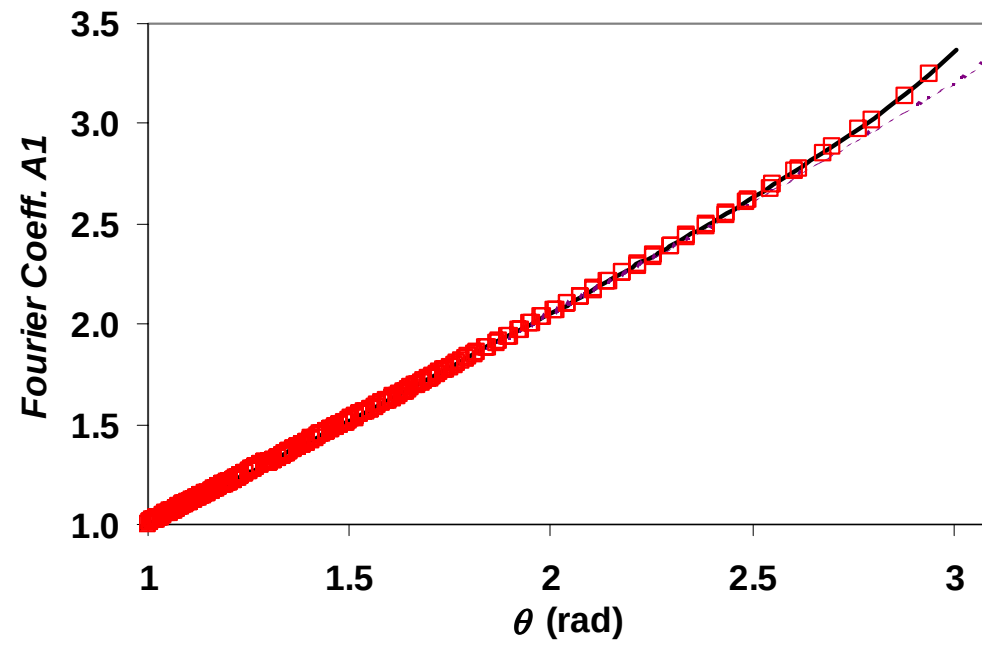


Fig. 4

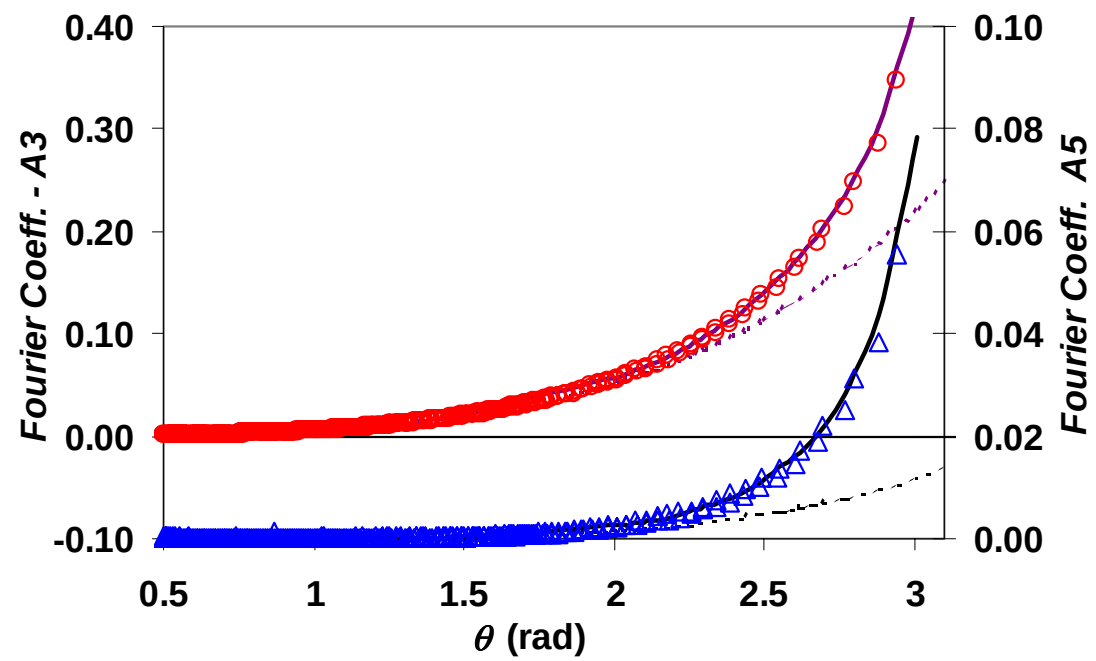


Fig. 5

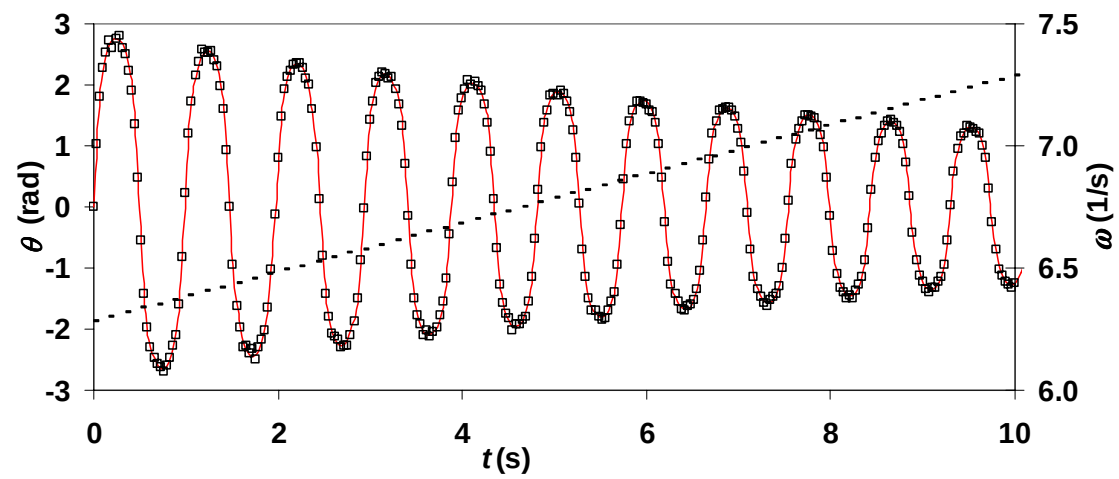


Fig. 6

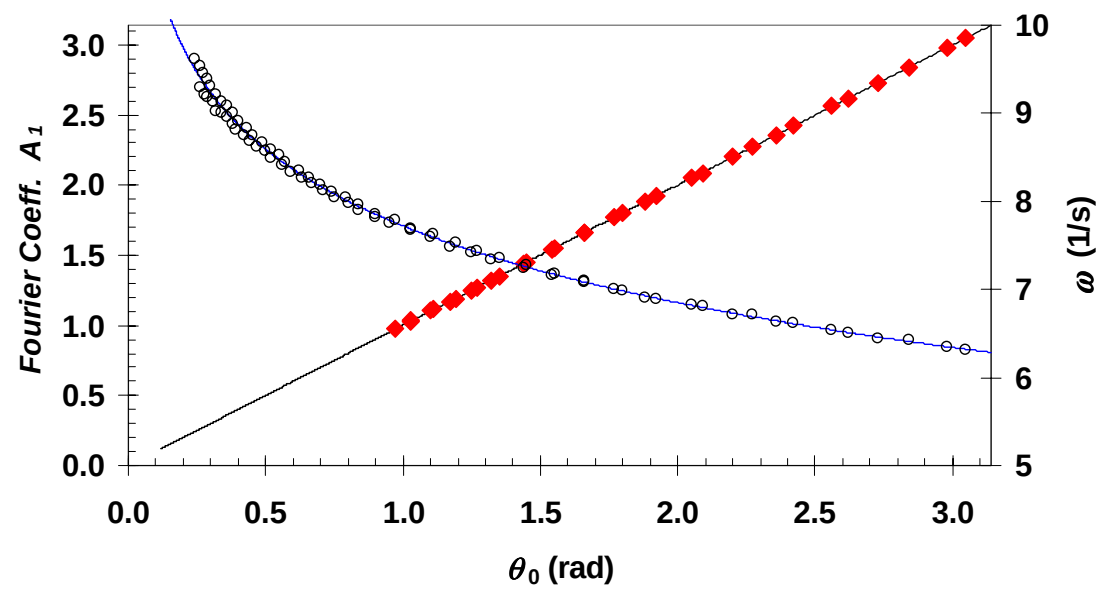


Fig. 7

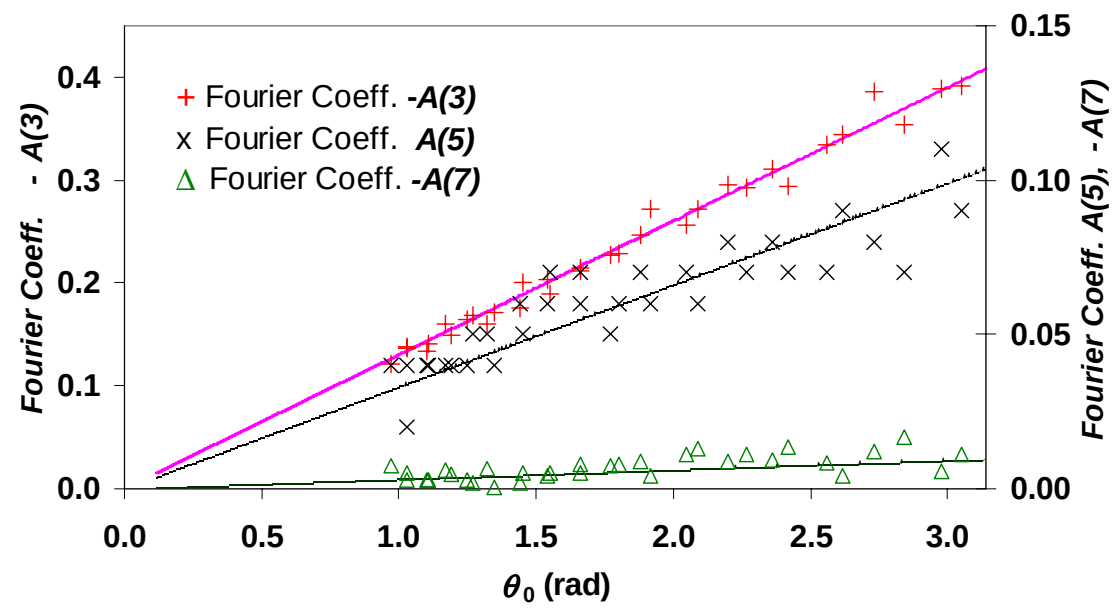


Fig. 8